

Remarks on parametric resonance of neutrino oscillations in the earth

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Abstract

Neutrino oscillations in matter can exhibit a specific resonance enhancement – parametric resonance, which is different from the MSW resonance. Recently it has been shown that the oscillations of atmospheric and solar neutrinos inside the earth can undergo parametric enhancement when neutrino trajectories cross the core of the earth. In this paper we continue the study of the parametric resonance of neutrino oscillations in the earth. The following issues are discussed: stability of the resonance with respect to the variations of the zenith angle of the neutrino source; higher-order resonances; prospects of the experimental observation of the parametric resonance of neutrino oscillations. We also comment on a recent controversy regarding the physical nature of the resonance enhancement of the oscillations of the core crossing neutrinos in the earth.

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1 Introduction

It is well known that neutrino oscillations in matter can differ significantly from oscillations in vacuum, the best studied example being the Mikheyev-Smirnov-Wolfenstein (MSW) effect [1, 2]. The MSW effect is an enhancement of the neutrino flavor transition probability due to an amplification of the oscillation *amplitude* in matter. However, possible enhancement effects of matter on neutrino oscillations are not limited to the MSW effect. It has been pointed out in [3, 4] that the probability of neutrino flavor transition can also be strongly enhanced if the oscillation *phase* undergoes certain modification in matter. This can happen if the change of the matter density along the neutrino path is correlated in a certain way with the change of the oscillation phase. This amplification of the neutrino oscillation probability in matter due to specific phase relationships has an interesting property that it can accumulate if the matter density profile along the neutrino path repeats itself, *i.e.* is periodic. The phenomenon is analogous to the resonance in dynamical systems whose parameters periodically vary with time – parametric resonance. It was therefore called parametric resonance of neutrino oscillations [3, 4]. In particular, in [4] the parametric resonance was considered for neutrino oscillations in a matter with a periodic step function (“castle wall”) density profile, which allows an exact analytic solution.

The parametric resonance can lead to large probabilities of neutrino flavor transition in matter even if the mixing angles both in vacuum and in matter are small. This happens because each half-wave oscillation of the transition probability is placed on the top of the previous one, *i.e.* the transition probability “drifts up”. If mixing angle in matter is very small (matter density is far from the MSW resonance density), the parametric resonance enhancement of neutrino oscillations can manifest itself only if the neutrinos pass through a large number of periods of density modulation, *i.e.* travel a sufficiently long distance. However, if matter density is not very far from the MSW resonance one, an interesting interplay between the MSW effect and parametric effects can occur. In particular, a strong parametric enhancement of neutrino oscillations can take place even if the neutrinos pass only through 1 - 2 periods of density modulation [5].

Although the parametric resonance of neutrino oscillations is certainly an interesting physical phenomenon, it requires that very special conditions be satisfied. Unfortunately, these conditions cannot be created in the laboratory because this would require either too long a baseline or neutrino propagation in a matter of too high a density (see Sec. 5 below). Until recently it was also unclear whether a natural object exists where these conditions can be satisfied for any known source of neutrinos. This situation has changed with a very important observation by Liu and Smirnov [6] (see also [7]), who have shown that the parametric resonance conditions can be approximately satisfied for the oscillations of atmospheric ν_μ into sterile neutrinos ν_s inside the earth.

It is known that the earth consists of two main structures – the mantle and the core. Within the mantle and within the core the matter density changes rather slowly (the density

variation scale is large compared to the typical oscillation lengths of atmospheric and solar neutrinos), but at their border it jumps sharply by about a factor of two. Therefore to a good approximation one can consider the mantle and the core as structures of constant densities equal to the corresponding average densities (two-layer model). Neutrinos coming to the detector from the lower hemisphere at zenith angles Θ in the range defined by $\cos \Theta = (-1) \div (-0.837)$ traverse the earth's mantle, core and then again mantle. Therefore such neutrinos experience a periodic “castle wall” potential, and their oscillations can be parametrically enhanced. Even though the neutrinos pass only through “1.5 periods” of density modulations (this would be exactly one period and a half if the distances neutrinos travel in the mantle and in the core were equal), the parametric effects on neutrino oscillations in the earth can be quite strong. Subsequently, it has been pointed out by Petcov [8] that the parametric resonance conditions can also be satisfied (and to even a better accuracy) for the $\nu_2 \leftrightarrow \nu_e$ oscillations in the earth in the case of the $\nu_e - \nu_{\mu(\tau)}$ mixing¹. This, in particular, may have important implications for the solar neutrino problem. The parametric resonance in the oscillations of solar and atmospheric neutrinos in the earth was further explored in a number of papers [9, 10, 11]. In the present paper we continue the study of this phenomenon. The following issues are addressed: the stability of the resonance with respect to the variation of the zenith angle of neutrino trajectory (Sec. 3), parametric resonances of higher order (Sec. 4), and the prospects of experimental observation of the parametric resonance in neutrino oscillations (Sec. 5). In the last section we comment on a recent controversy regarding the physical nature of the resonance enhancement of the oscillations of the core crossing neutrinos in the earth.

2 Parametric resonance in neutrino oscillations in the earth

In this section we shall summarize the main formulas describing neutrino oscillations in the earth which will be relevant for the subsequent discussion. We will be essentially following the formalism and notation of ref. [9].

Consider the oscillations in a 2-flavor neutrino system in a matter with periodic step function density profile. We will be assuming that one period of density modulation consists of two parts of the lengths R_m and R_c , with the corresponding effective matter densities N_m and N_c (“castle wall” density profile). For the $\nu_e \leftrightarrow \nu_{\mu(\tau)}$ oscillations the effective matter density coincides with the electron number density, whereas for the $\nu_{e,\mu,\tau} \leftrightarrow \nu_s$ oscillations in an isotopically symmetric matter it is a factor of two smaller. The parametric resonance in such a system occurs when the oscillations phases $2\phi_m$ and $2\phi_c$ acquired over the intervals

¹In [8] a different name for this phenomenon was suggested – neutrino oscillation length resonance. We prefer to follow the original terminology of [3, 4].

R_m and R_c are odd integer multiples of π [4, 6, 7]. Let us denote

$$\delta = \frac{\Delta m^2}{4E}, \quad V_i = \frac{G_F}{\sqrt{2}} N_i, \quad \omega_i = \sqrt{(\cos 2\theta_0 \delta - V_i)^2 + (\sin 2\theta_0 \delta)^2} \quad (i = m, c). \quad (1)$$

Here E , Δm^2 and θ_0 are the neutrino energy, mass squared difference and vacuum mixing angle, respectively. The difference of the neutrino eigenenergies in a matter of density N_i is $2\omega_i$, so that the oscillations phases acquired over the intervals R_m and R_c are

$$2\phi_m = 2\omega_m R_m, \quad 2\phi_c = 2\omega_c R_c. \quad (2)$$

The parametric resonance conditions can be written as [3, 4, 5, 6, 7, 8, 9]²

$$\phi_1 = \frac{\pi}{2} + k\pi, \quad \phi_2 = \frac{\pi}{2} + k'\pi, \quad k, k' = 0, 1, 2, \dots \quad (3)$$

The evolution of a system of oscillating neutrinos is conveniently described by the evolution matrix U . In the case of the 2-flavor system U is a 2×2 unitary matrix which can be parametrized as

$$U = Z - i\boldsymbol{\sigma}\mathbf{W}, \quad Z^2 + \mathbf{W}^2 = 1. \quad (4)$$

Here $\boldsymbol{\sigma}$ are the Pauli matrices in the flavor space. The evolution matrix U describes the evolution of an arbitrary initial state and therefore contains all the information relevant for neutrino oscillations. In particular, the probabilities of the neutrino flavor oscillations P and of $\nu_2 \leftrightarrow \nu_e$ oscillations P_{2e} are given by [9]

$$P = W_1^2 + W_2^2, \quad P_{2e} = \sin^2 \theta_0 + W_1(W_1 \cos 2\theta_0 + W_3 \sin 2\theta_0). \quad (5)$$

We shall consider neutrino oscillations in the earth in the two-layer approximation. Neutrinos coming to the detector from the lower hemisphere of the earth at zenith angles Θ in the range $\cos \Theta = (-1) \div (-0.837)$ (nadir angle $\Theta_n \equiv 180^\circ - \Theta \leq 33.17^\circ$) traverse the earth's mantle, core and then again mantle, *i.e.* three layers of constant density with the third layer being identical to the first one. In this case the parameters Z , \mathbf{W} take a very simple form [9]:

$$Z = 2 \cos \phi_m Y - \cos \phi_c, \quad (6)$$

$$\mathbf{W} = (2 \sin \phi_m \sin 2\theta_m Y + \sin \phi_c \sin 2\theta_c, \ 0, \ -(2 \sin \phi_m \cos 2\theta_m Y + \sin \phi_c \cos 2\theta_c)). \quad (7)$$

Here θ_m and θ_c are the neutrino mixing angles in the matter of the density N_m and N_c (*i.e.* in the mantle and in the core of the earth),

$$Y = \cos \phi_m \cos \phi_c - \sin \phi_m \sin \phi_c \cos(2\theta_m - 2\theta_c), \quad (8)$$

²In refs. [3, 4, 5] these conditions were derived for the particular case $k = k'$, which, however, includes the most important principal resonance with $k = k' = 0$.

and the vector \mathbf{W} was written in components. At the parametric resonance, *i.e.* when the conditions (3) are satisfied, the neutrino flavor transition probability takes the value [6, 7],

$$P = \sin^2(2\theta_c - 4\theta_m), \quad (9)$$

whereas the probability of the $\nu_2 \leftrightarrow \nu_e$ transitions is [8]

$$P_{2e} = \sin^2(2\theta_c - 4\theta_m + \theta_0). \quad (10)$$

These probabilities can be close to unity (the arguments of the sines close to $\pi/2$) even if the amplitudes of neutrino oscillations in the mantle, $\sin^2 2\theta_m$, and in the core, $\sin^2 2\theta_c$, are rather small. This can happen if the neutrino energy lies in the range $E_c < E < E_m$, where E_m and E_c are the values of the energy that correspond to the MSW resonance in the mantle and in the core of the earth. The probability P_{2e} is relevant for the description of the oscillations of solar neutrinos in the earth [12, 13]. In the case of small mixing angle MSW solution of the solar neutrino problem, $\sin^2 2\theta_0 < 10^{-2}$ [14], and P_{2e} practically coincides with P unless both probabilities are very small.

The trajectories of neutrinos traversing the earth are determined by their nadir angle Θ_n . The distances R_m and R_c that neutrinos travel in the mantle (each layer) and in the core are given by

$$R_m = R \left(\cos \Theta_n - \sqrt{r^2/R^2 - \sin^2 \Theta_n} \right), \quad R_c = 2R \sqrt{r^2/R^2 - \sin^2 \Theta_n}. \quad (11)$$

Here $R = 6371$ km is the earth's radius and $r = 3486$ km is the radius of the core. The matter density in the mantle of the earth ranges from 2.7 g/cm^3 at the surface to 5.5 g/cm^3 at the bottom, and that in the core ranges from 9.9 to 12.5 g/cm^3 (see, *e.g.*, [15]). The electron number fraction Y_e is close to $1/2$ both in the mantle and in the core. Taking the average matter densities in the mantle and core to be 4.5 and 11.5 g/cm^3 respectively, one finds for the $\nu_e \leftrightarrow \nu_{\mu,\tau}$ oscillations involving only active neutrinos the following values of V_m and V_c : $V_m = 8.58 \times 10^{-14} \text{ eV}$, $V_c = 2.19 \times 10^{-13} \text{ eV}$. For transitions involving sterile neutrinos $\nu_e \leftrightarrow \nu_s$ and $\nu_{\mu,\tau} \leftrightarrow \nu_s$, these parameters are a factor of two smaller.

3 Stability of the parametric resonance conditions

If the parametric resonance conditions (3) are satisfied, strong parametric enhancement of the oscillations of core crossing neutrinos in the earth can occur [6, 7, 8, 9, 10, 11]³. We shall now discuss these conditions. The phases ϕ_m and ϕ_c depend on the neutrino parameters

³For the parametric resonance to be a maximum of the transition probability, an additional condition has to be satisfied (two conditions in the case of $\nu_2 \leftrightarrow \nu_e$ oscillations [8]). For the oscillations between neutrinos of different flavor, this condition can be written as $\cos(2\theta_c - 4\theta_m) < 0$. For small vacuum mixing angles θ_0 it essentially reduces to $\delta < V_c$, with a small region around $\delta = V_m$ excluded.

Δm^2 , θ_0 and E and also on the distances R_m and R_c that the neutrinos travel in the mantle and in the core. The path lengths R_m and R_c vary with the nadir angle; however, as can be seen from (11), their changes are correlated and they cannot take arbitrary values. Therefore if for some values of the neutrino parameters a value of the nadir angle Θ_n exists for which, for example, the first condition in eq. (3) is satisfied, it is not obvious if at the same value of Θ_n the second condition will be satisfied as well. In other words, it is not clear if the parametric resonance conditions can be fulfilled for neutrino oscillations in the earth for at least one set of the neutrino parameters Δm^2 , θ_0 and E . However, as was shown in [8, 9], not only the parametric resonance conditions are satisfied (or approximately satisfied) for a rather wide range of the nadir angles covering the earth's core, they are fulfilled for the ranges of neutrino parameters which are of interest for the neutrino oscillations solutions of the solar and atmospheric neutrino problems. In particular, the conditions for the principal resonance ($k = k' = 0$) are satisfied to a good accuracy for $\sin^2 2\theta_0 \lesssim 0.1$, $\delta \simeq (1.1 \div 1.9) \times 10^{-13} \text{ eV}^2$, which includes the ranges relevant for the small mixing angle MSW solution of the solar neutrino problem and for the subdominant $\nu_\mu \leftrightarrow \nu_e$ and $\nu_e \leftrightarrow \nu_\tau$ oscillations of atmospheric neutrinos.

The fact that the parametric resonance conditions can be satisfied so well for neutrino oscillations in the earth is rather surprising. It is a consequence of a number of remarkable numerical coincidences. It has been known for some time [16, 6, 17] that the potentials V_m and V_c corresponding to the matter densities in the mantle and core, the inverse radius of the earth R^{-1} , and typical values of $\delta \equiv \Delta m^2/4E$ of interest for solar and atmospheric neutrinos, are all of the same order of magnitude – $(3 \times 10^{-14} - 3 \times 10^{-13}) \text{ eV}$. It is this surprising coincidence that makes appreciable earth effects on the oscillations of solar and atmospheric neutrinos possible. However, for the parametric resonance to take place, a coincidence by an order of magnitude is not sufficient: the conditions (3) have to be satisfied at least within a 50% accuracy [9]. This is exactly what takes place. In addition, in a wide range of the nadir angles Θ_n , with changing Θ_n the value of the parameter δ at which the resonance conditions (3) are satisfied slightly changes, but the fulfillment of these conditions is not destroyed. We shall show now that in this row of mysterious coincidences, at least the last one – the stability of the parametric resonance conditions with respect to variations of the nadir angle – has a simple explanation.

We begin with simple qualitative arguments. Consider the principal parametric resonance for which $k = k' = 0$. In figs. 1 and 2 the flavor transition probability P and the parameters $\phi_m - \pi/2$ and $\phi_c - \pi/2$ are plotted as functions of δ for $\sin^2 2\theta_0 = 0.01$ and $\Theta_n = 27.3^\circ$. The highest peak of P corresponds to $\delta \simeq 1.55 \times 10^{-13} \text{ eV}$ at which, as can be seen in fig. 2, both $\phi_m - \pi/2$ and $\phi_c - \pi/2$ nearly cross the abscissa, *i.e.* the conditions (3) are approximately satisfied. This is the parametric resonance peak; the smaller peaks on its left and right are due to the MSW resonances in the mantle and in the core, respectively. The minima of the curves $\phi_m - \pi/2$ and $\phi_c - \pi/2$ correspond to the minima of ω_m and ω_c . The values of R_m and R_c determine the steepness of the curves. With increasing nadir angle, R_m increases and R_c decreases, which means that the branches of the $\phi_m - \pi/2$ curve

become steeper and those of the $\phi_c - \pi/2$ curve become less steep. Increasing the steepness of $\phi_m - \pi/2$ would push the intersection point of the $\phi_m - \pi/2$ and $\phi_c - \pi/2$ curves upwards; however, decreasing the steepness of $\phi_c - \pi/2$ would push this intersection point in the opposite direction. As a result, in a wide range of the nadir angles Θ_n , with changing Θ_n the intersection point shifts along the abscissa but remains close to it, *i.e.* the parametric resonance conditions are not destroyed. It is easy to see that this only happens when the intersection takes place between the minima of $\phi_m - \pi/2$ and $\phi_c - \pi/2$, *i.e.* for δ in the range

$$V_m / \cos 2\theta_0 < \delta < V_m / \cos 2\theta_0. \quad (12)$$

If the intersection takes place outside this interval, the changes of the steepness of the $\phi_m - \pi/2$ and $\phi_c - \pi/2$ curves due to changing Θ_n will push their intersection point in the same direction, and this will quickly destroy the parametric resonance conditions. Therefore outside the range (12) the parametric resonance can only occur in rather narrow intervals of Θ_n . Interestingly, the interval (12) is not only favored due to the wide range of allowed values of Θ_n , but also corresponds to the region where the strongest parametric enhancement of neutrino oscillations takes place [8, 9].

We shall now quantify the above arguments. Each of the two parametric resonance conditions in (3) can be solved with respect to the value δ_0 of the parameter δ at the resonance:

$$\delta_0 = \cos 2\theta_0 V_m \pm \sqrt{[(2k+1)\pi]^2 / 4R_m^2 - \sin^2 2\theta_0 V_m^2} \quad (13)$$

$$\delta_0 = \cos 2\theta_0 V_c \pm \sqrt{[(2k'+1)\pi]^2 / 4R_c^2 - \sin^2 2\theta_0 V_c^2} \quad (14)$$

The parametric resonance can occur if the values of δ_0 given by these two expressions coincide or nearly coincide. Notice that the existence of the radicals in (13) and (14) in general puts upper limits on the vacuum mixing angle θ_0 . In particular, for the principal resonance, assuming that the resonance condition can be satisfied for all neutrino trajectories that cross the core (including the vertical ones), from (14) one finds [9]

$$\sin^2 2\theta_0 \leq \frac{\pi^2}{4(R_c)_{max}^2 V_c^2} \simeq 0.04. \quad (15)$$

If one excludes small nadir angles, the constraint becomes less stringent. For example, for $\sin^2 \Theta_n \geq 0.12$ one obtains $\sin^2 2\theta_0 \leq 0.07$. The analogous upper bound following from (13) are less restrictive except for the trajectories close to the edge of the core of the earth.

The (\pm) signs in eqs. (13) and (14) correspond to different regimes of the parametric resonance. We shall now choose the plus sign in (13) and the minus sign in (14), which for small θ_0 corresponds to the interval (12). The other cases can be considered similarly ⁴.

⁴In the cases that correspond to the two plus signs, the parametric resonance does not lead to a maximum of the transition probability; for two minus signs there exist parametric maxima, but they are typically smaller than in the region of δ that we are considering.

The consistency of eqs. (13) and (14) requires

$$\cos 2\theta_0 (V_c - V_m) = \sqrt{\frac{[(2k+1)\pi]^2}{4R_m^2} - \sin^2 2\theta_0 V_m^2} + \sqrt{\frac{[(2k'+1)\pi]^2}{4R_c^2} - \sin^2 2\theta_0 V_c^2}. \quad (16)$$

For small vacuum mixing angles θ_0 this simplifies to ⁵

$$(V_c - V_m) = \frac{\pi}{2} \left(\frac{2k+1}{R_m} + \frac{2k'+1}{R_c} \right). \quad (17)$$

Eq. (16) was obtained from the requirement that both conditions in (3) are satisfied at the same value of δ . If it is fulfilled, with changing Θ_n the resonance value of δ changes, but the parametric resonance conditions are still satisfied. Thus, it is the condition (16) that ensures the stability of the parametric resonance conditions with respect to the variations of the nadir angle. It remains now to check if or when this condition is indeed satisfied. We shall do that in the limit of the small θ_0 when the simplified form (17) applies. Similar analysis can be performed with the use of eq. (16).

The l.h.s. of eq. (17) is Θ_n independent, so must be the r.h.s. As we have pointed out above, with increasing Θ_n the path length R_m increases whereas R_c decreases, which is just a consequence of the spherical geometry of the earth. One can therefore expect at least a partial cancellation of the changes of the $(2k+1)/R_m$ and $(2k'+1)/R_c$ terms in their sum on the r.h.s. of eq. (17). We shall show now that for the principal parametric resonance with $k = k' = 0$ this cancellation is indeed quite good in a rather wide range of Θ_n . In fig. 3 the nadir angle dependence of R_m^{-1} , R_c^{-1} and $(R_m^{-1} + R_c^{-1})$ is plotted. One can see that while R_m^{-1} and R_c^{-1} change noticeably with Θ_n , their sum remains almost constant in the interval $\Theta_n = 0 \div 0.4$. For larger values of Θ_n this sum increases, but the increase does not exceed 25% for $\Theta_n \lesssim 0.54$. Thus, if the parametric resonance conditions are satisfied at some value of Θ_n in the interval $0 \leq \Theta_n \lesssim 0.54$, they will be approximately satisfied in the whole interval. This explains the stability of the parametric resonance conditions. The fact that they *are* satisfied to a good accuracy for at least one value in the above-mentioned interval is a consequence of the numerical coincidences that we discussed earlier.

4 Higher-order parametric resonances

Eq. (16) can be used for a systematic search for the parametric resonances of neutrino oscillations in the earth. Once a solution is found (*i.e.* the values of R_m and R_c are determined), one can find the corresponding value of δ_0 from (13) or (14). The properties of the principal resonance have been discussed in some detail in [6, 7, 8, 9]; here we shall concentrate on the higher-order resonances.

⁵For the particular case $k = k' = 0$ this condition was obtained in [8].

The resonances of interest are located in or close to the region (12). The resonances that are far from that region either do not correspond to maxima or lead to small values of the transition probability. In addition, they can exist only in rather narrow intervals of the nadir angle. We have been able to identify two higher-order resonances of the $\nu_e \leftrightarrow \nu_{\mu(\tau)}$ oscillations. Let us discuss their properties and possible implications.

(i) The resonance with $k = 0$, $k' = 1$. The resonance enhancement of the transition probability occurs in the interval of the nadir angles $\Theta_n \simeq 20^\circ \div 25^\circ$ and for relatively large vacuum mixing angles, $\sin^2 2\theta_0 \simeq 0.68 \div 0.86$. It corresponds to $\delta_0 \simeq (7 \div 9) \times 10^{-14}$ eV, depending on the values of Θ_n and $\sin^2 2\theta_0$. For typical values of these parameters the resonance is illustrated by figs. 4 and 5. One can see that there is practically no detuning, *i.e.* the conditions (3) are satisfied very well. The parametric peak at $\delta_0 \simeq 9 \times 10^{-14}$ eV is high and very wide, the energy width at half height being $\Delta E/E_0 \sim 1$ (the energies E_1 and E_2 at the low-energy and high-energy borders of the half-height region satisfy $E_2/E_1 \gtrsim 3$). The parametric resonance conditions can also be fulfilled for $\sin^2 2\theta_0 < 0.68$, but in this region the resonance corresponds to a low-lying saddle point rather than to a maximum of the transition probability. The existence of the resonance with $k = 0$, $k' = 1$ was pointed out in [11], but its properties were not discussed there.

(ii) The resonance with $k = 0$, $k' = 2$. This resonance occurs for small values of the nadir angle, $\Theta_n \lesssim 12^\circ$, and large (close to maximal) vacuum mixing, $\sin^2 2\theta_0 \gtrsim 0.9$. The resonance values of δ lie in the range $\delta_0 \simeq (0.65 \div 1.1) \times 10^{-13}$ eV. With increasing $\sin^2 2\theta_0$ the resonance value of δ decreases. This resonance is shown for $\Theta_n = 0$ and $\sin^2 2\theta_0 = 0.95$ in figs. 6 and 7. The parametric peak at $\delta_0 \simeq 1 \times 10^{-13}$ eV is slightly less wide than in the case (i): $E_2/E_1 \simeq 2.1$. One can see from fig. 6 that the curves $\phi_m - \pi/2$ and $\phi_c - 5\pi/2$ cross the abscissa at different but close points which are located symmetrically around $\delta = 1 \times 10^{-13}$ eV. The detuning at $\delta = 1 \times 10^{-13}$ eV does not exceed ± 0.15 , *i.e.* is rather small.

What are possible implications of these higher-order resonances? Unlike the principal resonance which can occur for both small and large values of θ_0 , the resonances (i) and (ii) are only possible if the vacuum mixing angle is rather large, $\sin^2 2\theta_0 \gtrsim 0.7$. For atmospheric neutrinos, for values of Δm^2 of practical interest, such large mixing in the $\nu_e - \nu_\mu$ system is excluded by the Super-Kamiokande [18] and CHOOZ [19] data. Large mixing angle is in principle of interest for the vacuum oscillations and large mixing angle MSW solutions of the solar neutrino problem. However, the former requires very small values of δ , orders of magnitude below those at which the parametric resonance of the neutrino oscillations in the earth can occur. The large mixing angle MSW solution, on the contrary, requires too large values of δ . For $\Delta m^2 \simeq 7 \times 10^{-6}$ eV² which is the smallest value allowed by the total event rates of the four solar neutrino experiments [14], and $E = 10$ MeV which is a typical boron neutrino energy, one gets $\delta \simeq 1.75 \times 10^{-13}$ eV. This is about a factor of 1.6 - 2.5 larger than the values of δ_0 that are necessary for the higher-order parametric resonances discussed above. These resonances could, however, occur for *hep* neutrinos which constitute the highest-energy component of the solar neutrino flux (endpoint energy 18.8

MeV). Smaller values of Δm^2 (and, consequently, of δ) are allowed if the boron neutrino flux is considered as a free parameter [14, 20]. Relatively small values of Δm^2 are also allowed in 3-flavor analyses of the solar neutrino data, even when the constraints on the mixing matrix element U_{e3} following from the CHOOZ experiments are taken into account [21, 20]. Notice, however, that although the values $\Delta m^2 \simeq 7 \times 10^{-6} \text{ eV}^2$ or even less are allowed at large vacuum mixing angles by the total event rates, they are disfavored by the recent Super-Kamiokande data on the zenith angle distribution of the solar neutrino events.

We conclude that the higher-order parametric resonances are less interesting than the principal one. The domain of the neutrino parameters in which they can occur for the oscillations of solar neutrino in the earth is currently disfavored by the solar neutrino data, though not completely ruled out. These resonances are of no interest for the atmospheric neutrino oscillations. It would be interesting to look for other possible neutrino sources for which these resonances could play a role.

5 Can the parametric resonance of neutrino oscillations be observed?

Besides being an interesting physical phenomenon, the parametric resonance of neutrino oscillations can provide us with an important additional information about neutrino properties. Therefore experimental observation of this effect would be of considerable interest. We shall now discuss the prospects for experimental observation of the parametric resonance of neutrino oscillations in the earth, having in mind mainly the principal resonance. There are two main sources of neutrinos for which the parametric resonance can be important – atmospheric neutrinos and solar neutrinos. Both sources have their advantages and disadvantages from the point of view of the possibility of observation of the parametric resonance. We shall now briefly discuss them.

We start with atmospheric neutrinos. The parametric resonance can occur in the $\nu_\mu \leftrightarrow \nu_s$ [6, 7] and also in the subdominant $\nu_e \leftrightarrow \nu_{\mu(\tau)}$ channels of oscillations [10]. It can affect the distributions of μ -like events and also (in the case of the $\nu_e \leftrightarrow \nu_{\mu(\tau)}$ oscillations) lead to interesting peculiarities in the zenith angle distributions of the multi-GeV e-like events.

The observation of the parametric effects is hampered by the loose correlation between the directions of the momenta of atmospheric neutrinos and of the charged leptons which they produce and which are actually detected. Because of this the trajectories of neutrinos coming to the detector are not known very precisely. In addition, the data are presented for certain samples of events (sub-GeV, multi-GeV, upward through-going, upward stopping) which includes collecting data over rather wide energy intervals. The contributions of the parametric peaks may therefore be integrated over together with other possible enhancement peaks – due to the MSW resonances in the mantle and in the core, making the distinction between these effects difficult. Also, strong resonance enhancement effects (both parametric

and MSW) can only occur either for neutrinos or for antineutrinos, depending on the sign of Δm^2 ⁶. The present atmospheric neutrino experiments do not distinguish between neutrinos and antineutrinos, therefore possible matter effects are “diluted” in the sum of the ν - and $\bar{\nu}$ -induced events. At certain values of the ratio of the muon and electron neutrino fluxes $r(E, \Theta_n)$ depending on the value of mixing angle θ_{23} the parametric effects on e-like events are suppressed [10].

Atmospheric neutrinos have some advantages for observation of the parametric resonance of neutrino oscillations in the earth. Neutrinos come to the detectors from all directions, which means that practically the whole solid angle covering the earth’s core will contribute to the effect. There are no additional suppression factors due to a specific composition of the incoming neutrino flux which may quench the earth’s matter effect on the oscillations of solar neutrinos (see below). Parametric effects may provide a unique probe of the neutrino mixing angle θ_{13} with sensitivity going far beyond that of the long-baseline accelerator and reactor experiments [10, 22]. Possible ways of improving the prospects of the experimental observation of the parametric effects in the atmospheric neutrino oscillations include using various energy cuts, finer zenith angle binning and detectors capable of detecting the recoil nucleon, which would enable one to reconstruct the direction of an incoming neutrino [10, 22].

Solar neutrinos can experience a strong parametric enhancement of their oscillations in the earth if the small mixing angle MSW effect is the correct explanation of the the solar neutrino deficit [8, 9]. The parametric enhancement can occur in a wide range of values of $\sin^2 2\theta_0$ and for the nadir angles Θ_n almost completely covering the core of the earth. The trajectory of each detected neutrino is exactly known. For boron neutrinos the resonance occurs at the values of Δm^2 which correspond to the central part of the allowed interval for the small mixing angle MSW solution of the solar neutrino problem.

However, there are some disadvantages, too. Unfortunately, due to their geographical location, the existing solar neutrino detectors have a relatively low time during which solar neutrinos pass through the core of the earth to reach the detector every calendar year. The Super-Kamiokande detector has a largest fractional core coverage time equal to 7%. In [16] it was suggested to build a new detector close to the equator in order to increase the sensitivity to the earth regeneration effect; this would also maximize the parametric resonance effects in oscillations of solar neutrinos in the earth. In the case of the MSW solutions of the solar neutrino problem the probability P_{SE} of finding a solar ν_e after it traverses the earth depends sensitively on the average ν_e survival probability in the sun \bar{P}_S [12, 13]:

$$P_{SE} = \bar{P}_S + \frac{1 - 2\bar{P}_S}{\cos 2\theta_0} (P_{2e} - \sin^2 \theta_0). \quad (18)$$

The probability P_{2e} can experience a strong parametric enhancement, but in the case of

⁶In general, the parametric enhancement can take place even for “wrong sign” Δm^2 , but in the case of neutrino oscillations in the earth these effects are small.

small mixing angle MSW solution of the solar neutrino problem the probability \bar{P}_S for the Super-Kamiokande and SNO experiments turns out to be rather close to $1/2$. This means that the effects of passage through the earth on solar neutrinos should be strongly suppressed. The current best fit of the solar neutrino data is not far from the line in the parameter space where \bar{P}_S is exactly equal to $1/2$ and $P_{SE} = \bar{P}_S$ (*i.e.* the earth matter effects are absent). Whether or not it will be possible to observe the parametric resonance of the oscillations of solar neutrinos in the earth depends on how close to this line the true values of $\sin^2 2\theta_0$ and Δm^2 are. By now the Super-Kamiokande experiment has not observed, within its experimental accuracy, any enhancement of neutrino signal for earth core crossing neutrinos [23]. This can be because the parametric enhancement of the neutrino oscillations in the earth does not occur (*e.g.* if the true solution of the solar neutrino problem is vacuum oscillations or large mixing angle MSW effect), or because the values of Δm^2 and $\sin^2 2\theta_0$ are too close to those at which $\bar{P}_S = 1/2$. Hopefully, with accumulated statistics of the Super-Kamiokande and forthcoming data from the SNO experiment the situation will soon be clarified.

It is interesting to note that the Super-Kamiokande data on the zenith angle dependence of the solar neutrino events seems to indicate some deficiency of the events due to the core-crossing neutrinos rather than an excess [23], although it is not statistically significant. Should this deficiency be confirmed by future data with better statistics, it could have a natural explanation in terms of the parametric resonance of neutrino oscillations. As follows from (18), the parametric enhancement of P_{2e} for core crossing neutrinos can lead to a deficiency of the events if the neutrino parameters are in the small- $\sin^2 2\theta_0$ part of the allowed region which corresponds to $P_S > 1/2$ (see, *e.g.*, fig. 10 in ref. [24]). In this case one should also have an “opposite sign” overall day-night effect (fewer events during the night than during the day). In any case, given the current experimental constraints on the neutrino parameters, if the small mixing angle MSW effect is the true solution of the solar neutrino problem, the only hope to observe earth matter (day-night) effects on solar neutrinos seems to be through the parametric resonance of oscillations of core crossing neutrinos.

As we have seen, observing the parametric resonance in oscillations of solar and atmospheric neutrinos in the earth is not an easy task. Can one create the necessary matter density profile and observe the parametric resonance in neutrino oscillations in the laboratory (*i.e.* short-baseline) experiments? Unfortunately, the answer to this question seems to be negative. The parametric resonance can occur when the mean oscillation length in matter approximately coincides with the matter density modulation length [3, 4, 5]: $l_m \simeq L$. In a matter of density N_i the oscillation length is given by $l_m = \pi/\omega_i$ where ω_i was defined in (1). Let us require $l_m \lesssim 1$ km. Assume first that $V_i \gtrsim \delta$, *i.e.* ω_i are dominated by matter density terms. Then for $l_m \lesssim 1$ km one would need a matter of mass density $\rho_i \geq 3.3 \times 10^4$ g/cm³, clearly not a feasible value. Conversely, for $\rho_i \leq 10$ g/cm³, one finds $l_m \gtrsim 3300$ km, a distance comparable with the earth’s radius. Consider now the opposite case, $\delta \gg V_i$. Then the oscillation length in matter essentially coincides with the vacuum

oscillations length which in principle can be rather short provided that the vacuum mixing angle θ_0 is small (otherwise this would contradict reactor and accelerator data). However, in this case there is another problem. Requiring $l_m \lesssim 1$ km one finds $\delta \gtrsim 2.5 \times 10^{-10}$ eV. For $\rho_i \lesssim 10$ g/cm³ one therefore has $V_i/\delta \lesssim 10^{-3}$. This means that the mixing angles in matter are very close to the vacuum one, $\theta_i \simeq \theta_0(1 + V_i/\delta)$, and so their difference is very small: $\Delta\theta = \theta_2 - \theta_1 \simeq (\Delta V/\delta)\theta_0 \lesssim 10^{-3}\theta_0$. When the difference of mixing angles in matter is small, the parametric effects can manifest themselves only if neutrinos travel over a large number of periods, $n \simeq \pi/4\Delta\theta$ [3, 4, 5, 9]. Therefore in this case the necessary baseline is $\sim \pi^2/(4\theta_0\Delta V) \gtrsim 3 \times 10^3$ km, again too large. One can conclude that the sole presently known object where the parametric resonance of neutrino oscillations can take place is the earth, as was first pointed out in [6, 7].

6 Parametric resonance or oscillation length resonance?

Recently, there have been a considerable controversy regarding the physical nature of the resonance enhancement of the oscillations of the core crossing neutrinos in the earth. In refs. [6, 7, 9, 25, 26, 10] this enhancement was interpreted as a parametric resonance effect. However, in refs. [8, 27, 28, 11] it was interpreted as a new effect, called neutrino oscillation length resonance. It was argued that this effects has nothing to do with the parametric resonance of neutrino oscillations. This controversy has led to a considerable confusion and we therefore believe that it is worthwhile to discuss the issue in some detail. We shall argue here that the resonance enhancement of oscillations of core crossing neutrinos is a typical parametric resonance effect.

The general possibility of the parametric resonance in neutrino oscillations was suggested independently in [3] and [4] and further studied in [5]. In these papers it was shown that the parametric resonance is a specific resonance enhancement of neutrino transition probability which occurs due to a certain relationship between the density modulation scale and neutrino oscillation length, does not require large amplitude of neutrino oscillations, and has a property to accumulate when the matter density profile repeats itself, *i.e.* is periodic. The resonance enhancement of the oscillations of core crossing neutrinos in the earth fully matches this description. The conditions (11) in [8] (reproduced also in [27, 28, 11, 29]), which are claimed to be the conditions for the oscillation length resonance, coincide with the parametric resonance conditions given in [3, 4, 5, 6, 7] (see eqs. (10) and (17) in [4], eqs. (2) and (4) in [5]), and in particular eq. (7) in [6] and eq. (12) in [7] where the parametric resonance conditions were formulated specifically for the case of the oscillations of core crossing neutrinos in the earth). Moreover, the probability of the neutrino flavor transition at the oscillation length resonance given in [11] coincides with the oscillation probability at the parametric resonance given in [6, 7] (see eq. (9) above).

We shall now comment on the arguments presented in refs. [8, 27, 28, 29] (and given below in *italics*) in support of the point of view that the resonance enhancement of the

oscillations of core crossing neutrinos in the earth is very different from the parametric resonance effects in the case of the “castle wall” matter density profile studied in [4].

- In [4] the results were presented for the specific case of small densities ($V_{m,c} \ll \delta$) which does not correspond to the situation with oscillations of neutrinos in the earth.

The case $V_{m,c} \ll \delta$ was discussed in ref. [4] just in order to stress that the proposed effect – parametric resonance in neutrino oscillations – is very different from the MSW effect. The parametric resonance can take place for any relation between the matter-induced potentials and δ (see, e.g., [5]).

- In [4] the case of the density profile with equal spatial dimensions of the layers of constant density ($R_m = R_c$) was studied, whereas in the earth $R_m \neq R_c$.

The nature of the effect does not depend on whether R_m and R_c are equal to each other or not. In fact, there is a value of the nadir angle ($\Theta_n = 27.3^\circ$) which determines the neutrino trajectory through the earth along which $R_m = R_c$. Fig. 1 was drawn for this particular value of the nadir angle; it shows a rather typical pattern of the resonance enhancement in the oscillations of core crossing neutrinos (compare with analogous figures in refs. [8, 9, 10, 11] ⁷).

- The density profile along the trajectory of core crossing neutrinos is not periodic even in the two-layer approximation; neutrino path length falls short even of 1.5 periods.

We shall first comment on the second part of this argument. In general, it is incorrect: neutrino path length is less than 1.5 periods of density modulation for $\cos \Theta_n > 0.89$ but exceeds this value for $0.837 < \cos \Theta_n < 0.89$, which constitutes about 31% of the solid angle covering the earth’s core. This, however, is unimportant. The term “1.5 periods” was used in refs. [9, 10] just as a shorthand for describing the situation when neutrinos cross three layers of constant density with the third layer being identical to the first one.

In fig. 8 we show the coordinate dependence of the probability of neutrino oscillations in the earth for $\Theta_n = 11.5^\circ$ (solid curve), and for comparison also show the probability in the hypothetical case when neutrinos propagate through the full two periods of density modulation (dashed curve). Obviously, in both cases the physical nature of the enhancement of the transition probability is the same: the probability drifts up and each new half-wave piece of the oscillating curve is placed on the top of the previous one. This pattern is typical of the parametric resonance of neutrino oscillations [3, 4, 9].

Now about the periodicity of the matter density profile of the earth in the two-layer approximation. In the strict mathematical sense, a function $f(x)$ can be called periodic only if it is defined in the infinite interval $x \in (-\infty, \infty)$. It was shown, however, in [3, 4] that the complete flavor conversion is possible even if neutrinos travel over a finite distance

⁷Similar figures can also be found in some earlier papers [30] the authors of which have found the parametric resonance of neutrino oscillations in the earth in numerical calculations but have not recognized its parametric nature.

in a matter of periodic density. In [5] it was demonstrated that this distance can be quite short (1-2-3 periods) if the parametric resonance energy is not very far from the MSW resonance energy. This is exactly what happens in the case of the oscillations of the core crossing neutrinos in the earth. The term “periodic” was used in refs. [3, 4, 5, 9] in this limited sense – as a finite piece of the periodic density profile. Since the path length of core crossing neutrinos in the earth exceeds one period of density modulation $R_m + R_c$, the density profile of the earth in the two-layer approximation satisfies this definition.

Actually, the only relevant question here is if it is really important that the third layer traversed by the neutrinos going through the earth is identical to the first one. This brings us to the next objection.

- *The specific enhancement of the neutrino transition probability under discussion can take place even if neutrinos cross a number of layers of constant but different densities.*

In general, this is correct. Assume, for example, that the neutrinos cross three layers of densities $N_1 < N_2 < N_3$, and that we are free to choose the spatial extensions of these layers, so that the condition similar to (3) can be satisfied. Then, in principle, one can have a strong enhancement of the transition probability. However, the resonance conditions (3) are just the extremality conditions; for the resonance to be a maximum of the transition probability, additional conditions have to be satisfied. In the case of neutrino flavor oscillations in a step-function density profile with $N_3 = N_1$, there is only one such condition; in the case $N_1 < N_2 < N_3$ there are three additional conditions. Thus, in this latter case there are altogether six conditions (three conditions on oscillation phases and three additional conditions), whereas in the case $N_3 = N_1$ there are only three conditions.

As was discussed above, the possibility of the resonance enhancement of the oscillations of the core crossing neutrinos in the earth is a result of a number of remarkable numerical coincidences. It is because of these coincidences that all three conditions for the parametric resonance of neutrino oscillations can be satisfied. It is much less probable that *six* different conditions could be satisfied as a result of some coincidences of parameters. Fortunately, in the case of the oscillations of the core crossing neutrinos in the earth, the third layer that the neutrinos cross *is* identical to the first one, *i.e.* the matter density profile *is* periodic and so there is a chance that the resonance enhancement of neutrino oscillations under the discussion will be observed.

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Figure captions

Fig. 1. Transition probability P for $\nu_e \leftrightarrow \nu_{\mu,\tau}$ oscillations in the earth for core-crossing neutrinos vs δ . $\sin^2 2\theta_0 = 0.01$, $\Theta_n = 27.3^\circ$.

Fig. 2. Phases $\phi_m - \pi/2$ (dashed curve) and $\phi_c - \pi/2$ (solid curve) vs δ for $\nu_e \leftrightarrow \nu_{\mu,\tau}$ oscillations in the earth. $\sin^2 2\theta_0 = 0.01$, $\Theta_n = 27.3^\circ$.

Fig. 3. Parameters $1/R_m$ (long-dashed curve), $1/R_c$ (short-dashed curve) and $1/R_m + 1/R_c$ (solid curve) as functions of the nadir angle Θ_n . All distances are in the units of the earth's radius R ; Θ_n is in radians.

Fig. 4. Same as fig. 1 but for $\sin^2 2\theta_0 = 0.77$, $\Theta_n = 23.3^\circ$.

Fig. 5. Phases $\phi_m - \pi/2$ (solid curve) and $\phi_c - 3\pi/2$ (dashed curve) vs δ for $\nu_e \leftrightarrow \nu_{\mu,\tau}$ oscillations in the earth. $\sin^2 2\theta_0 = 0.77$, $\Theta_n = 23.3^\circ$.

Fig. 6. Same as fig. 1 but for $\sin^2 2\theta_0 = 0.95$, $\Theta_n = 0$.

Fig. 7. Phases $\phi_m - \pi/2$ (solid curve) and $\phi_c - 5\pi/2$ (dashed curve) vs δ for $\nu_e \leftrightarrow \nu_{\mu,\tau}$ oscillations in the earth. $\sin^2 2\theta_0 = 0.95$, $\Theta_n = 0$.

Fig. 8. Solid curve: transition probability P for $\nu_e \leftrightarrow \nu_{\mu,\tau}$ oscillations in the earth as a function of the distance t (measured in units of the earth's radius R) along the neutrino trajectory. $\delta = 1.8 \times 10^{-13}$ eV, $\sin^2 2\theta_0 = 0.01$, $\Theta_n = 11.5^\circ$. Dashed curve: the same for a hypothetical case of neutrino propagation over full two periods of density modulation ($t_{max} = 2(R_m + R_c)$).

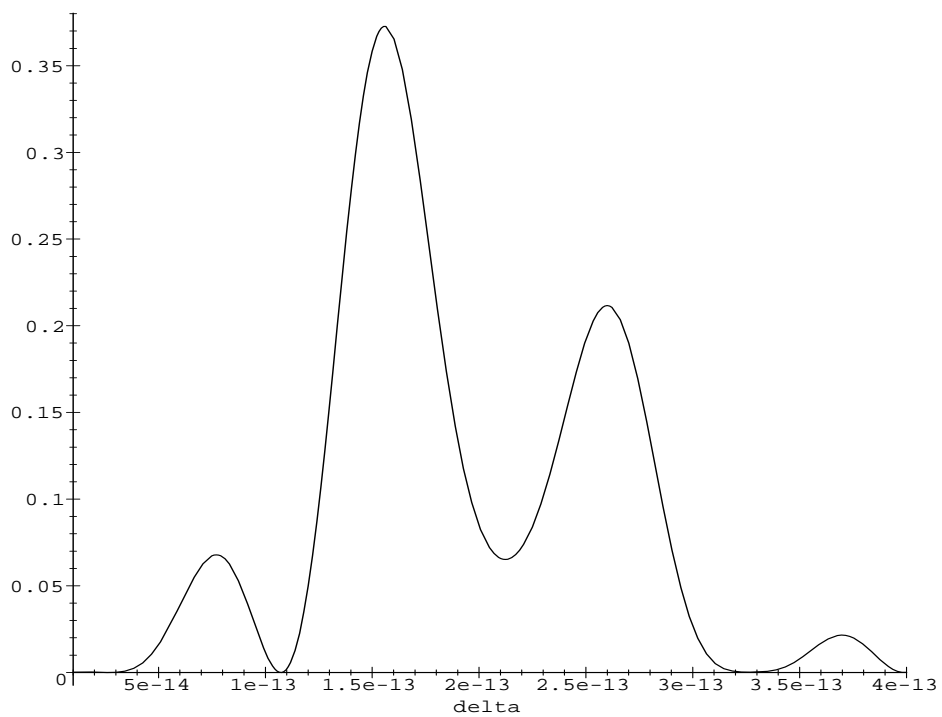


Fig. 1.

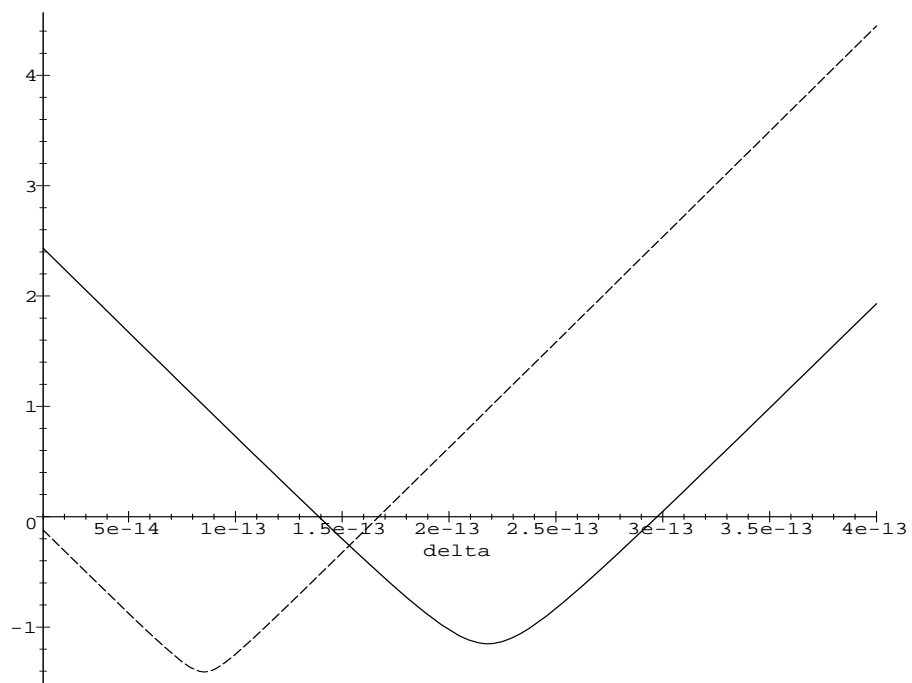


Fig. 2.

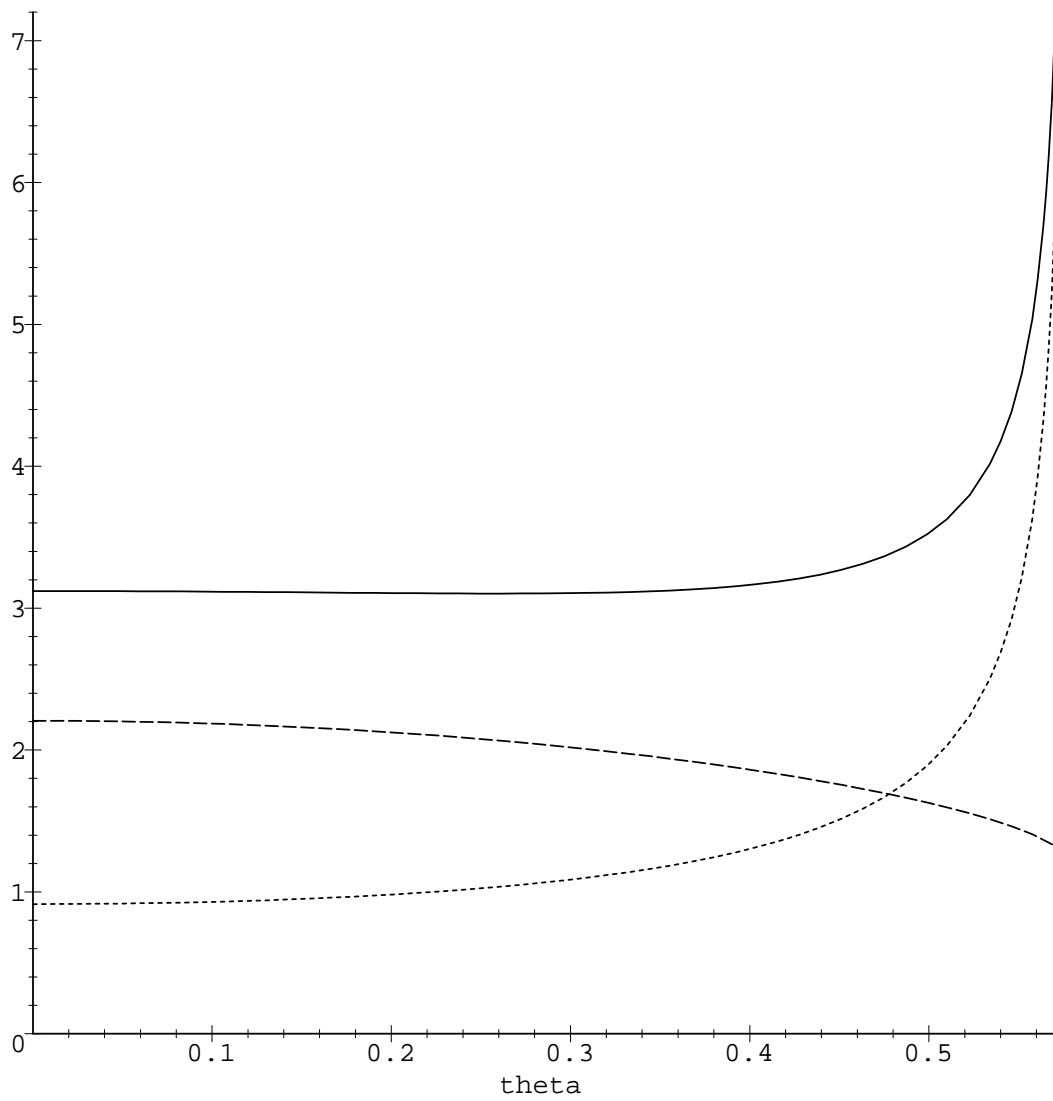


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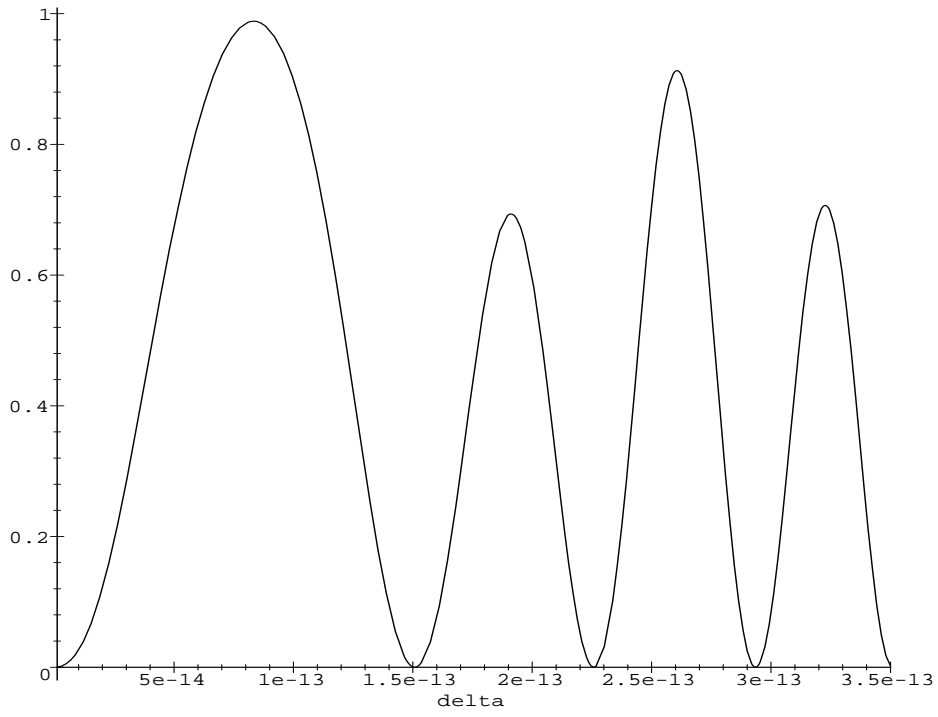


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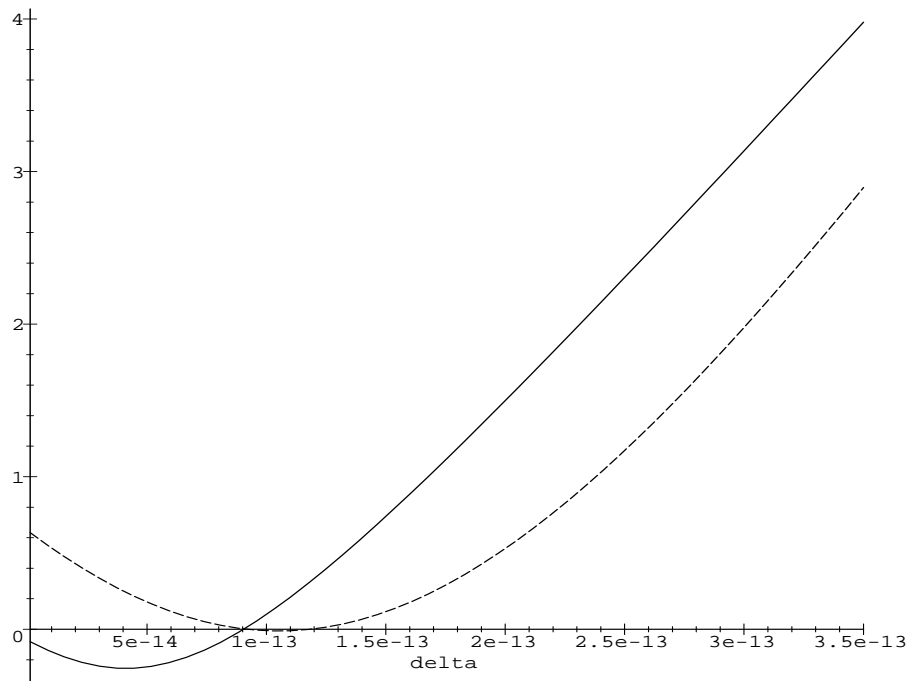


Fig. 5.

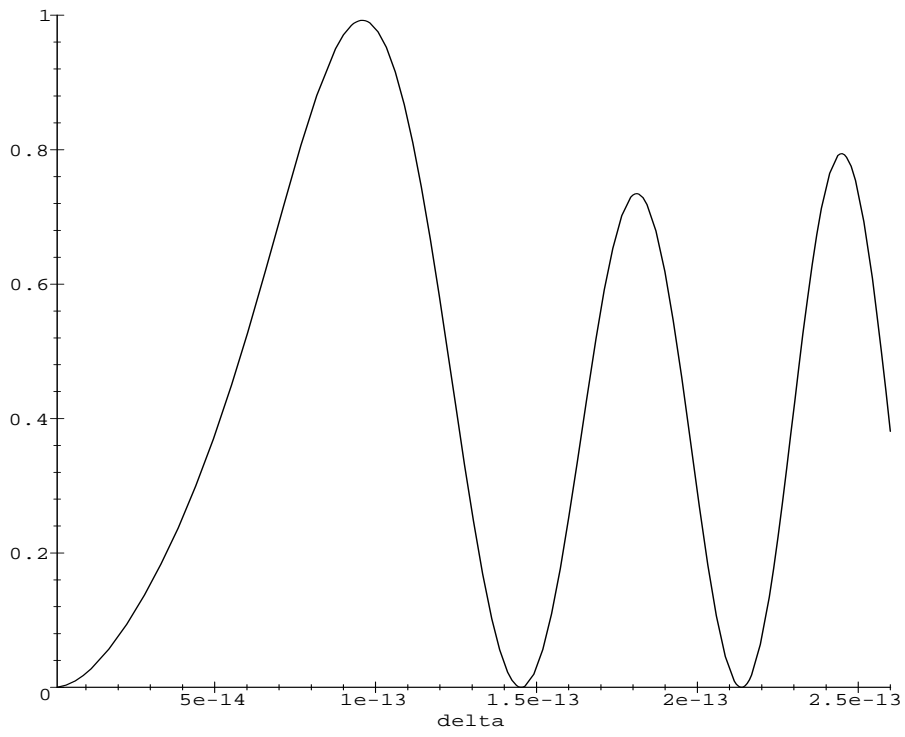


Fig. 6.

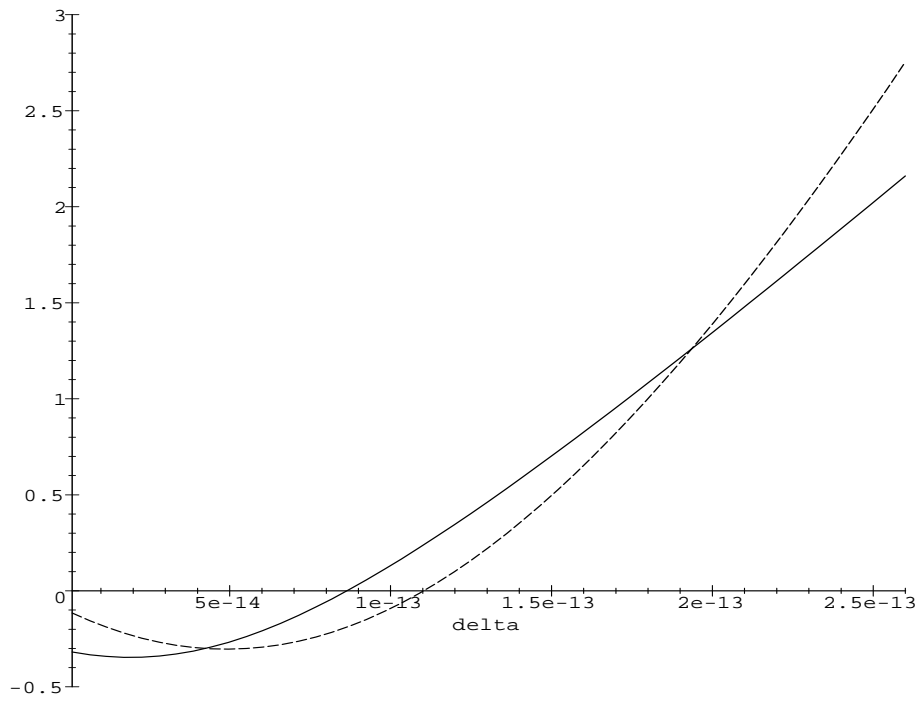


Fig. 7.

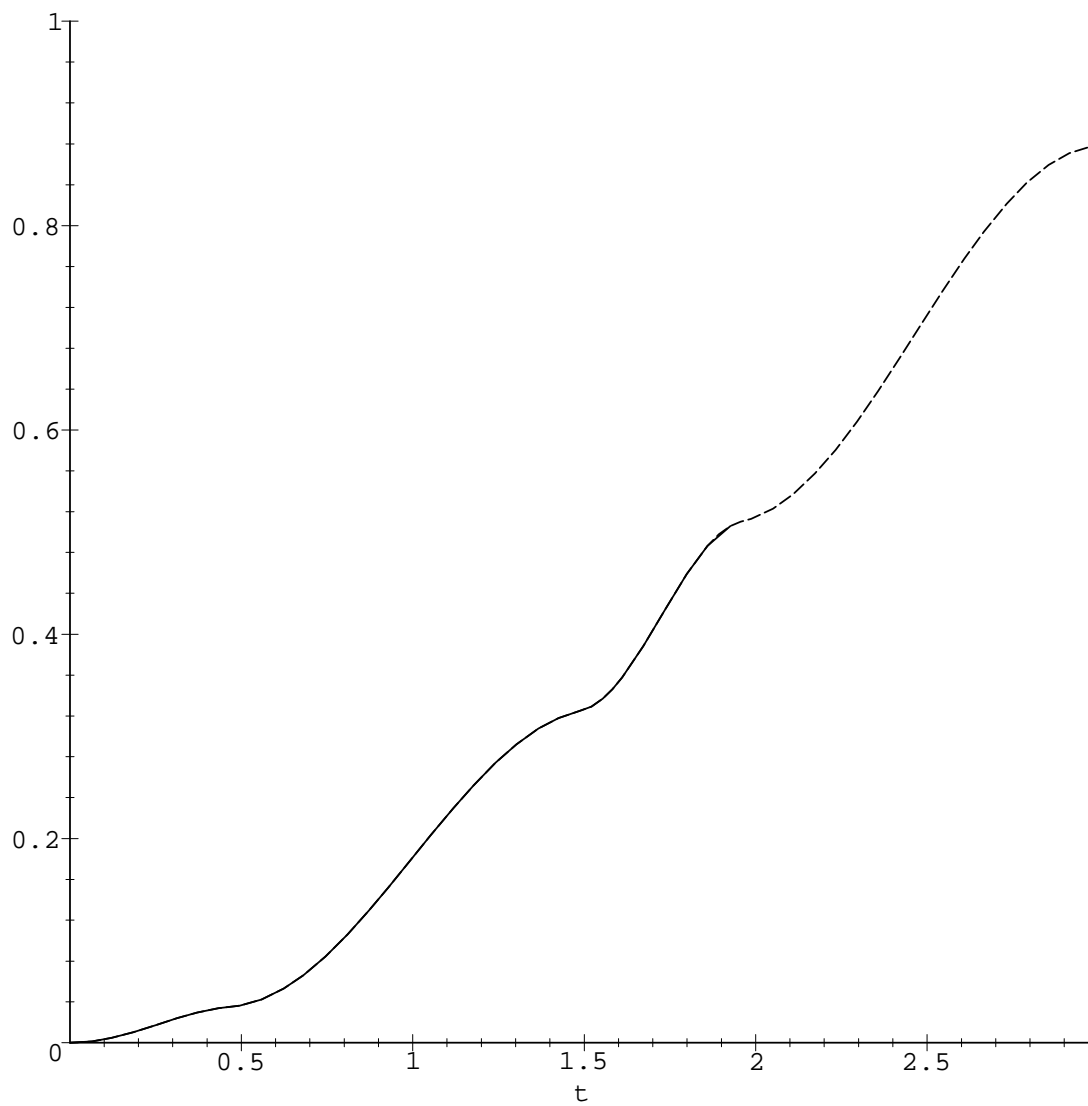


Fig. 8.